

A New Causal Ideal Internal Dynamics Generator

Quan Quan and Kai-Yuan Cai

Abstract

The design of ideal internal dynamics (IID) generators, namely solving IID, is a fundamental problem, which is a key step to handle the nonminimum-phase output tracking problem. In this paper, for a class of unstable matrix differential equations, a new causal dynamic IID generator is proposed, whose parameters are partly chosen via $\mathcal{H}_2/\mathcal{H}_\infty$ optimization. Compared with existing similar generators, it is applicable to matrix differential equations with singular system matrices and is easily extended to slowly time-varying matrix differential equations without extra computation.

Index Terms

Nonminimum-phase systems, ideal internal dynamics, causal case, tracking.

I. INTRODUCTION

A system is nonminimum-phase if its internal dynamics (ID) are unstable [1]. Nonminimum-phase output tracking is a challenging, real-life control problem that has been extensively studied. An important way for this problem is to identify the state references such that the output tracking problem can be converted to be an easier stabilization problem, which can be solved by using conventional control methods, such as sliding mode control methods [2],[3]. State references are composed of output references and internal state references. The former are often given, whereas the latter is difficult to obtain for an unstable ID, namely for a nonminimum-phase system. A bounded solution to the unstable ID is called the ideal internal dynamics (IID) [2]. A basic *IID Problem* can be stated as:

IID Problem: Given $\xi \in \mathcal{L}_\infty([0, \infty), \mathbb{R})^1$, $A \in \mathbb{R}^{n \times n}$ and $N \in \mathbb{R}^n$, find an initial condition η_0 such that the solution $\eta(t)$ to the following differential equation

$$\dot{\eta}(t) = A\eta(t) + N\xi(t), \eta(0) = \eta_0, t \geq 0 \quad (1)$$

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¹ $f \in \mathcal{L}_\infty([0, \infty), \mathbb{R}^n)$ denotes that $f(t) \in \mathbb{R}^n$ and $\sup_{t \geq 0} \|f(s)\| < \infty$.

belongs to $\mathcal{L}_\infty([0, \infty), \mathbb{R})$.

The *IID Problem* is in fact about the noncausal (offline) case, where $\xi(s)$, $s \in [0, \infty)$ is available before finding the solution η . If A is stable, then the IID can be obtained by solving the differential equation (1) directly in forward time, whereas it cannot for an unstable A . For an unstable A , the basic idea of solving the IID with an unstable A is to run the stable parts forward in time and the unstable parts backward with the priori information. However, it does not work in the causal (online) case, where only $\xi(s)$, $s \in [0, t]$ is available to determine the solution η at the time t . This problem can be formulated in general as:

Causal IID Problem: Given $\xi \in \mathcal{L}_\infty([0, T], \mathbb{R})$, $\hat{\eta}_T(0) = 0$, $A \in \mathbb{R}^{n \times n}$, $N \in \mathbb{R}^n$ and $\delta > 0$, find a differentiable function $\hat{\eta}_T \in \mathcal{L}_\infty([0, T], \mathbb{R}^n)$ such that²

$$\dot{\hat{\eta}}_T(T) - A\hat{\eta}_T(T) - N\xi(T) \rightarrow \mathcal{B}(\delta), \text{ as } T \rightarrow \infty.$$

In [2], the noncausal IID problem was considered for a class of forcing terms generated by a known nonlinear exosystem. The problem was further solved for a class of more general systems and a class of more general forcing terms in [4]. However, these inversion-based approaches require the entire output references ahead of time which restricts the use. To overcome this limitation, the preview-based stable-inversion approaches were proposed [5],[6]. It requires the finite-previewed (in time) future output reference and thus enables the online implementation. Such a problem can be formulated as a modified *Causal IID Problem* that finds a solution $\hat{\eta}_T \in \mathcal{L}_\infty([0, T], \mathbb{R}^n)$ by $\xi \in \mathcal{L}_\infty([0, T + T_{pre}], \mathbb{R})$, where $T_{pre} > 0$ is the preview time. It has been shown that a large enough preview time is critical to ensure the precision in the preview-based output tracking. However, for some cases, the forcing term $\xi(t)$ in (1) may be an online estimate of uncertainties, namely the future information is unavailable. Therefore, the solution idea for the noncausal IID is inapplicable to the causal IID problem. To the best of our knowledge, the solutions to the causal IID problem are only limited to a class of bounded forcing term generated by an exosystem. For a class of forcing term generated by a linear exosystem, the IID can be given exactly by solving a Sylvester equation proposed in [7]. For a nonlinear exosystem, we have to resort to a first-order partial differential equation proposed in [8]. The two resulting IID generators can generate the IID directly, which can be considered as static IID generators. However, they require full knowledge of the state of the exosystem, which however may not be obtained directly. Moreover, the resulting IID

² $\mathcal{B}(\delta) \triangleq \{\xi \in \mathbb{R} \mid \|\xi\| \leq \delta\}$, $\delta > 0$; the notation $x(t) \rightarrow \mathcal{B}(\delta)$ means $\inf_{y \in \mathcal{B}(\delta)} |x(t) - y| \rightarrow 0$.

will preserve the noise if the state of the exosystem is noisy. For these reasons, the authors suppose, a dynamic IID generator was proposed to solve the IID for the equation (1) in [3]. Furthermore, by using higher-order sliding mode differentiators, it was modified in [9] for an unknown matrix A . However, both dynamic generators do not cover the case that A is singular as they require obtaining A^{-1} . Furthermore, in the case of a time-varying matrix, they will be time-consuming. For example, if adopt $\frac{d}{dt}A^{-1}(t) = -A^{-1}(t)\frac{d}{dt}A(t)A^{-1}(t)$ ³ to generate $A^{-1}(t)$ online, then we have to calculate about n^2 differential equations. The same difficulty also exists in solving a time-varying Sylvester equation.

In this paper, we propose a new causal dynamic IID generator for a class of perturbed forcing terms generated by linear exosystems. Analysis shows that the equation (1) is solvable if A is singular under the conditions consistent with that for the Sylvester equation proposed in [7]. Furthermore, to suppress the perturbation by the noise, the parameters are partly chosen via $\mathcal{H}_2/\mathcal{H}_\infty$ optimization so that the error bound caused by the perturbation can be evaluated. To show the advantage, the proposed IID generator is also applied to a slowly time-varying unstable differential equation in the simulation. Compared with existing similar generators, it avoids computing A^{-1} so that it can cover the case that A is singular, and is further easier to apply to matrix differential equations with slowly time-varying system matrices. Moreover, the proposed dynamic IID generator only needs to calculate about n differential equations. This reduces the computational complexity. Finally, it should be pointed out that the proposed IID generator can also be applied to the tracking problem for nonlinear nonminimum-phase systems by following the idea as in [3],[9], i.e., to lump weakly nonlinear terms and uncertainties into the forcing term ξ .

II. PROBLEM FORMULATION AND PRELIMINARY RESULTS

A. Problem Formulation

Consider the following unstable matrix differential equation:

$$\dot{\eta} = A\eta + N\xi, \eta(0) = 0 \quad (2)$$

where

– $\eta \in \mathbb{R}^n$ is the state;

³ $A(t)A^{-1}(t) = I_n \Rightarrow \frac{d}{dt}A(t)A^{-1}(t) + A(t)\frac{d}{dt}A^{-1}(t) = 0_{n \times n} \Rightarrow \frac{d}{dt}A^{-1}(t) = -A^{-1}(t)\frac{d}{dt}A(t)A^{-1}(t)$

– $\xi \in \mathcal{L}_\infty([0, \infty), \mathbb{R})$ (it will be extended to be a vector later) could be modeled as follows:

$$\dot{w} = Sw, \xi = E^T w \quad (3)$$

where $w \in \mathbb{R}^m, S \in \mathbb{R}^{m \times m}, E \in \mathbb{R}^m$; here we consider the causal case, namely the signal $\xi(s), s \in [0, t]$ is available at the time $t > 0$.

– $N \in \mathbb{R}^n$, and $A \in \mathbb{R}^{n \times n}$ is a non-Hurwitz matrix.

Denote $\hat{\eta}$ to be the estimate. The objective is to obtain a bounded estimate $\hat{\eta}$ such that $y(t) = \dot{\hat{\eta}}(t) - A\hat{\eta}(t) - N\xi(t) \rightarrow 0$ as $t \rightarrow \infty$. Furthermore, consider the case that ξ is a vector.

Before proceeding further with the development of this work, the following preliminary result is needed.

Lemma 1. If and only if $\text{rank}(F - \lambda I_n) = n - 1$ for every eigenvalue $\lambda \in \mathbb{C}$ of $F \in \mathbb{R}^{n \times n}$, then there exists a vector $B \in \mathbb{R}^n$ such that the pair (F, B) is controllable⁴.

Proof. See Appendix A.

III. A NEW CAUSAL IDEAL INTERNAL DYNAMICS GENERATOR

Our IID generator is proposed as follows:

$$\dot{x} = A_{cl}x + N_{cl}\xi, x(0) = 0 \quad (4a)$$

$$\hat{\eta} = C_{cl}^T x \quad (4b)$$

where

$$x = \begin{bmatrix} v \\ \hat{\eta} \\ e \end{bmatrix} \in \mathbb{R}^{m+n+1}, v \in \mathbb{R}^m, \hat{\eta} \in \mathbb{R}^n, e \in \mathbb{R}, C_{cl} = \begin{bmatrix} 0_{m \times n} \\ I_n \\ 0_{1 \times n} \end{bmatrix} \in \mathbb{R}^{(m+n+1) \times n},$$

$$A_{cl} = \begin{bmatrix} S & 0_{m \times n} & L_{11} \\ 0_{n \times m} & A & L_{12} \\ L_{21} & L_{22} & L_3 \end{bmatrix} \in \mathbb{R}^{(m+n+1) \times (m+n+1)}, N_{cl} = \begin{bmatrix} 0_{m \times 1} \\ N \\ 0 \end{bmatrix} \in \mathbb{R}^{m+n+1},$$

$$L_{11} \in \mathbb{R}^m, L_{12} \in \mathbb{R}^n, L_{21} \in \mathbb{R}^{1 \times m}, L_{22} \in \mathbb{R}^{1 \times n}, L_3 \in \mathbb{R}.$$

⁴It should be noted that this useful property was first shown by Wonham [10]. Later, the proof was simplified by Antsaklis [11] in a completely different way. We have completed this proof based on some basic knowledge on matrix before knew these previous proofs. So, our proof is completely different from those in [10],[11].

The basic idea is to make (4) satisfy the following two conditions:

- i) A_{cl} is stable;
- ii) $e(t) \rightarrow 0$ as $t \rightarrow \infty$.

By taking ξ as the input and x as the state, the condition i) implies the bounded-input bounded-state stability of (4a), namely the resulting $\hat{\eta}$ is bounded. On the other hand, (4a) contains the dynamics $\dot{\hat{\eta}} = A\hat{\eta} + L_{12}e + N\xi$. So, the condition ii) implies that the resulting $\hat{\eta}$ satisfies the unstable matrix differential equation (2) asymptotically. Therefore, we achieve the proposed objective.

It is easy to satisfy the condition i) by choosing appropriate gains $L_{11}, L_{12}, L_{21}, L_{22}, L_3$. On the other hand, to satisfy the condition ii), we introduce the dynamics $\dot{v} = Sv + L_{11}e$ into (4), where the matrix S is the same as that in (3). The idea is inspired by a new viewpoint on the internal model principle proposed in [12]: e will vanish if it becomes an input of the internal model such as $\dot{v} = Sv + L_{11}e$, which is further incorporated into a stable closed-loop linear system. These results are stated in *Theorems 1-4*.

Theorem 1. For (4), suppose i) $\xi \in \mathcal{L}_\infty([0, \infty), \mathbb{R})$ is generated by (3); ii) the gains $L_{11}, L_{12}, L_{21}, L_{22}, L_3$ satisfy $\max \operatorname{Re} \lambda(A_{cl}) < 0$. Then $e \rightarrow 0$ as $t \rightarrow \infty$, meanwhile keeping x bounded. Furthermore, $y = \hat{\eta} - A\hat{\eta} - N\xi \rightarrow 0$ as $t \rightarrow \infty$.

Proof. See Appendix B.

The key condition of *Theorem 1* is to find the gains $L_{11}, L_{12}, L_{21}, L_{22}, L_3$ satisfying $\max \operatorname{Re} \lambda(A_{cl}) < 0$. However, a question immediately arises as to under what conditions such gains exist for given S and A . In *Theorem 2*, we will answer this question. Denote

$$A_S = \begin{bmatrix} S & 0_{m \times n} \\ 0_{n \times m} & A \end{bmatrix} \in \mathbb{R}^{(n+m) \times (n+m)}, L_1 = \begin{bmatrix} L_{11} \\ L_{12} \end{bmatrix} \in \mathbb{R}^{n+m}.$$

Theorem 2. If and only if $\operatorname{rank}(A_S - \lambda I_{n+m}) = n + m - 1$ for every eigenvalue $\lambda \in \mathbb{C}$ of A_S , then there exists a vector $L_1 \in \mathbb{R}^{m+n}$ such that the pair (A_S, L_1) is controllable. Furthermore, if matrix S and A have an eigenvalue in common, then the pair (A_S, L_1) is uncontrollable for any $L_1 \in \mathbb{R}^{m+n}$.

Proof. The first part of *Theorem 2* can be claimed by *Lemma 1* obviously. If matrix S and A have an eigenvalue in common, denoted by λ_c , then

$$\begin{aligned} \operatorname{rank}(A_S - \lambda_c I_{n+m}) &= \operatorname{rank}(S - \lambda_c I_n) + \operatorname{rank}(A - \lambda_c I_m) \\ &\leq m + n - 2. \end{aligned}$$

We can conclude this proof for the second part of *Theorem 2* by *Lemma 1*. \square

With *Theorems 1-2* in hand, we have

Theorem 3. For (4), suppose i) $\xi \in \mathcal{L}_\infty([0, \infty), \mathbb{R})$ is generated by (3) with appropriate initial values; ii) $\text{rank}(A_S - \lambda I) = m + n - 1$ for every eigenvalue λ of A_S . Then i) there must exist gains $L_{11}, L_{12}, L_{21}, L_{22}, L_3$ satisfying $\max \text{Re } \lambda(A_{cl}) < 0$; furthermore ii) $e \rightarrow 0$ as $t \rightarrow \infty$, meanwhile keeping $x(t)$ bounded. Moreover, $y = \dot{\hat{\eta}} - A\hat{\eta} - N\xi \rightarrow 0$ as $t \rightarrow \infty$.

Proof. See Appendix C.

Remark 1. The IID can be given exactly [7]: $\eta = \Pi w$, where $\Pi \in \mathbb{R}^{n \times m}$ satisfies the Sylvester equation $\Pi S = A\Pi + NE^T$. Such equation has a unique solution if and only if S and A have no eigenvalues in common [17, Theorem 13.18, p. 145]. It is easy to see that the following two conditions are equivalent:

$$S \text{ and } A \text{ have no eigenvalues in common} \Leftrightarrow \text{rank}(A_S - \lambda I_n) = n + m - 1.$$

Therefore, the solvability condition of the proposed generator is consistent with that of the Sylvester equation $\Pi S = A\Pi + NE^T$. If A is singular, then S cannot be singular to ensure the existence of the vector $L_1 \in \mathbb{R}^{n+m}$. Unlike the IID generators given by [3],[9], the proposed IID generator allows A to be singular in some cases. For example, the pair (A_S, L_1) is controllable with $A = 0, S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, L_1 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$. This feature broadens the application of the proposed IID generator.

Let us consider that ξ is a vector rather than a scalar, namely

$$\dot{\eta} = A\eta + \sum_{k=1}^l N_k \xi_k, \eta(0) = 0 \quad (5)$$

where $\eta \in \mathbb{R}^n, \xi_k \in \mathbb{R}, N_k \in \mathbb{R}^n, k = 1, \dots, l$. We have the following result:

Theorem 4. For (5), suppose i) $\xi_k \in \mathcal{L}_\infty([0, \infty), \mathbb{R})$ and can be generated by (3) with an appropriate initial value, $k = 1, \dots, l$; ii) $\text{rank}(A_S - \lambda I) = m + n - 1$ for every eigenvalue λ of A_S . Then i) there must exist gains $L_{11}, L_{12}, L_{21}, L_{22}, L_3$ satisfying $\max \text{Re } \lambda(A_{cl}) < 0$; ii) furthermore, the following IID generator

$$\begin{aligned} \dot{x} &= A_{cl}x + \sum_{k=1}^l N_{cl,k} \xi_k, x(0) = 0 \\ \hat{\eta} &= C_{cl}^T x \end{aligned} \quad (6)$$

can drive $y = \dot{\hat{\eta}} - A\hat{\eta} - \sum_{k=1}^l N_k \xi_k \rightarrow 0$ as $t \rightarrow \infty$, meanwhile keeping $x(t)$ bounded, where $x \in \mathbb{R}^{m+n+1}, \hat{\eta} \in \mathbb{R}^n, N_{cl,k} = \begin{bmatrix} 0_{1 \times m} & N_k^T & 0 \end{bmatrix}^T \in \mathbb{R}^{m+n+1}, A_{cl}, C_{cl}$ are same as in (4).

Proof. By the superposition principle or additive decomposition [13], the IID generator (6) can be decomposed into

$$\begin{aligned}\dot{x}_k &= A_{cl}x_k + N_{cl,k}\xi_k, x_k(0) = 0 \\ \hat{\eta}_k &= C_{cl}^T x_k, k = 1, \dots, l\end{aligned}\quad (7)$$

with the relation

$$x = \sum_{k=1}^l x_k, \hat{\eta} = \sum_{k=1}^l \hat{\eta}_k. \quad (8)$$

By conditions i)-ii) and *Theorem 3*, the IID generator (7) for each $\xi_k(t)$ can drive $y_k = \dot{\hat{\eta}}_k - A\hat{\eta}_k - N_k\xi_k \rightarrow 0$ as $t \rightarrow \infty$, meanwhile keeping $x_k(t)$ bounded. By (8), we have

$$y = \dot{\hat{\eta}} - A\hat{\eta} - \sum_{k=1}^l N_k\xi_k = \sum_{k=1}^l \left(\dot{\hat{\eta}}_k - A\hat{\eta}_k - N_k\xi_k \right) \rightarrow 0$$

as $t \rightarrow \infty$, meanwhile keeping $x(t) = \sum_{k=1}^l x_k(t)$ bounded. \square

IV. $\mathcal{H}_2/\mathcal{H}_\infty$ OPTIMAL DESIGN OF IID GENERATOR

So far, we have proposed the structure of the IID generators, and further investigated the existence of their parameters $L_{11}, L_{12}, L_{21}, L_{22}, L_3$. However, there exist infinite choices of the parameters $L_{11}, L_{12}, L_{21}, L_{22}, L_3$ to satisfy $\max \operatorname{Re} \lambda(A_{cl}) < 0$. In this section, we will design these parameters according to some optimization principles.

In practice, the forcing term ξ often cannot be modeled as (3) without perturbation. Assume $\varepsilon \in \mathbb{R}$ to be a bounded perturbation. Driven by $\xi + \varepsilon$, the solution to (4) satisfies

$$\begin{aligned}\dot{x}_\varepsilon &= A_{cl}x_\varepsilon + N_{cl}(\xi + \varepsilon), x_\varepsilon(0) = 0 \\ \hat{\eta}_\varepsilon &= C_{cl}^T x_\varepsilon.\end{aligned}\quad (9)$$

We expect to design the parameters $L_{11}, L_{12}, L_{21}, L_{22}, L_3$ such that $\hat{\eta}_\varepsilon - \hat{\eta}$ is not sensitive to the perturbation ε . Subtracting (4) from (9) results in

$$\begin{aligned}\dot{x}_e &= A_{cl}x_e + N_{cl}\varepsilon, x_e(0) = 0 \\ \hat{\eta}_e &= C_{cl}^T x_e\end{aligned}\quad (10)$$

where $\hat{\eta}_e = \hat{\eta}_\varepsilon - \hat{\eta}$ and $x_e = x_\varepsilon - x$. Denote

$$A'_S = \begin{bmatrix} A_S & L_1 \\ 0_{1 \times (m+n)} & 0 \end{bmatrix}, L_{23} = \begin{bmatrix} L_{21} & L_{22} & L_3 \end{bmatrix}^T.$$

Then $A_{cl} = A'_S + B_1 L_{23}^T$. The system (10) can be rewritten as

$$\begin{aligned}\dot{x}_e &= A'_S x_e + B_1 u + N_{cl} \varepsilon, x_e(0) = 0 \\ \hat{\eta}_e &= C_{cl}^T x_e \\ u &= L_{23}^T x_e\end{aligned}$$

which is shown in Fig.1.

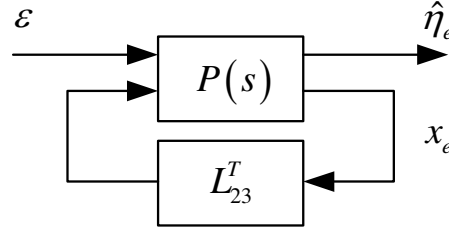


Fig. 1. State-feedback control

Although some tracking and robustness are best captured by an \mathcal{H}_∞ criterion, noise insensitivity is more naturally expressed by the \mathcal{H}_2 criterion. Robust pole placement specifications are also required for reasonable feedback gains. Denote by $T_{\hat{\eta}_e \varepsilon}$ the closed-loop transfer functions from ε to $\hat{\eta}_e$. For simplicity, we determine L_1 similar to (15) beforehand. Then, our goal is to design a state-feedback law $u = L_{23}^T x_e$ that

- Maintains $\|T_{\hat{\eta}_e \varepsilon}\|_\infty$ below some prescribed value $\gamma_0 > 0$.
- Maintains $\|T_{\hat{\eta}_e \varepsilon}\|_2$ below some prescribed value $\nu_0 > 0$.
- Minimizes an $\mathcal{H}_2/\mathcal{H}_\infty$ trade-off criterion of the form $\alpha \|T_{\hat{\eta}_e \varepsilon}\|_\infty + \beta \|T_{\hat{\eta}_e \varepsilon}\|_2$, $\alpha \geq 0, \beta \geq 0$.
- Places the closed-loop poles in a prescribed region \mathcal{D} of the open left-half plane.

Formally, the objective is to find L_{23} such that:

$$\begin{aligned}\min_{L_{23}} \quad & \alpha \|T_{\hat{\eta}_e \varepsilon}\|_\infty + \beta \|T_{\hat{\eta}_e \varepsilon}\|_2 \\ \text{s.t.} \quad & \|T_{\hat{\eta}_e \varepsilon}\|_\infty < \gamma_0 \\ & \|T_{\hat{\eta}_e \varepsilon}\|_2 < \nu_0 \\ & \lambda(A_{cl}) \in \mathcal{D} = \{z \in \mathbb{C} \mid Q + Mz + M\bar{z} < 0\}\end{aligned} \tag{11}$$

where matrices $Q = Q^T$ and M is a suitable matrix.

Remark 2. The perturbation is not necessary to be $\varepsilon \in \mathcal{L}_2$ in practice although \mathcal{H}_2 optimization is considered. From (4), the state is still bounded if ε is bounded and $\max \text{Re } \lambda(A_{cl}) < 0$.

Remark 3. The MATLAB function “msfsyn”[15] is applicable to solve the optimization problem (11).

V. SIMULATION EXAMPLES

For simplicity, in the following examples, the prescribed region \mathcal{D} of the open left-half plane is chosen to be an intersection of a conic sector centered at the origin with inner angle $\frac{3\pi}{4}$ and a vertical strip $[-10, -1]$, shown in Fig.2.

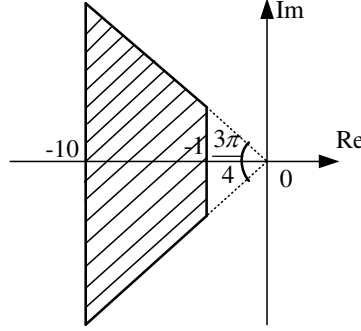


Fig. 2. Prescribed region \mathcal{D}

Example 1. In (2), $A = 0, N = 1$, where ξ is generated by (3) with $S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $E = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $w(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Since A_S has three different eigenvalues $0, \pm j$, $\text{rank}(A_S - \lambda I) = 2$ for $\lambda = 0, \pm j$. Obviously, the dynamic IID generators proposed in [3] and [9] are inapplicable to this example. Similar to (15) in *Appendix A*, L_1 is chosen as $L_1 = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^T$. Choosing $\gamma_0 = 20, \nu_0 = 20, \alpha = 0.5, \beta = 0.5$ and solving (11) by the MATLAB function “msfsyn”, we obtain $L_{23} = 10^3 \times \begin{bmatrix} 0.5360 & 1.0746 & -0.9743 & -0.0219 \end{bmatrix}^T$ with $\|T_{\hat{\eta}_e \varepsilon}\|_\infty = 1.75$ and $\|T_{\hat{\eta}_e \varepsilon}\|_2 = 2.61$. By solving (4) in forward time, the IID is obtained. As shown in Fig.3, the one-dimensional estimated IID $\hat{\eta}$ is bounded and $y = \dot{\hat{\eta}} - A\hat{\eta} - N\xi \rightarrow 0$ as $t \rightarrow \infty$. Moreover, it is easy to see that the estimated IID $\hat{\eta}$ converges to the desired IID. In the presence of ε , as shown in Fig.4, it is easy to see that the estimated IID can also converge to the desired IID with a small error.

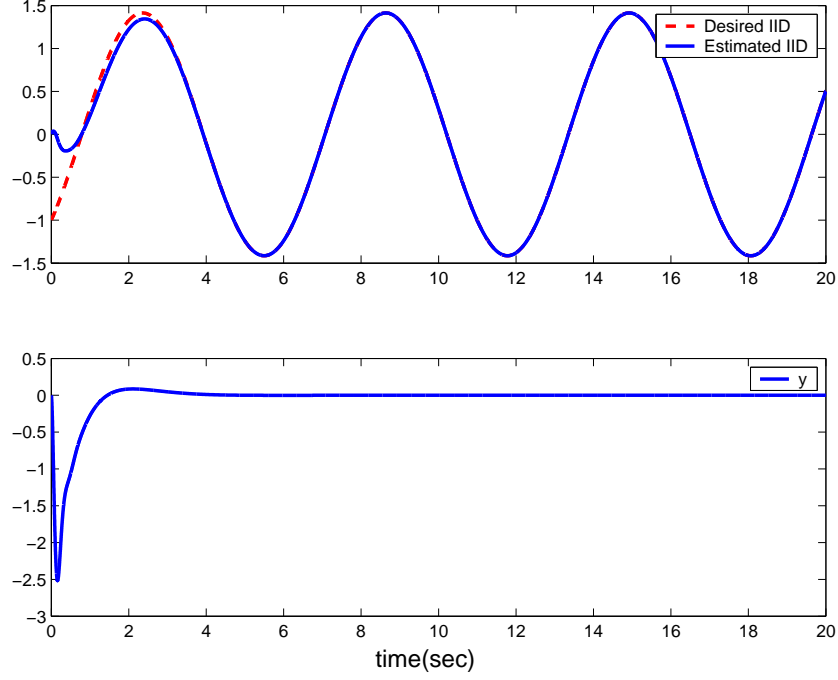


Fig. 3. Output of the IID generator in *Example 1*

Example 2. In (5),

$$A(t) = \begin{bmatrix} 0.2 \sin(0.05t) & 1 \\ -1 & 1 \end{bmatrix}, N_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, N_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, l = 2 \quad (12)$$

where ξ_1, ξ_2 are generated respectively by (3) with

$$S = \begin{bmatrix} 0 & 0.2 \\ -0.2 & 0 \end{bmatrix}, E = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, w_1(0) = \begin{bmatrix} 1 & 0 \end{bmatrix}^T, w_2(0) = \begin{bmatrix} 0 & 1 \end{bmatrix}^T.$$

In the IID generator (6), A will be replaced by $A(t)$ in (12) to obtain an approximate IID. However, for sake of designing L_1 and L_{23} , we consider $A(t) \equiv \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$ first. Similar to (15) in *Appendix A*, L_1 is designed as $L_1 = \begin{bmatrix} 1 & 0 & 1 & 0.0670 \end{bmatrix}^T$. Choosing $\gamma_0 = 20, \nu_0 = 20, \alpha = 0.5, \beta = 0.5$ and solving (11) by the MATLAB function “msfsyn”, we obtain $L_{23} = 10^4 \times \begin{bmatrix} -0.5702 & 1.0009 & 0.5159 & 0.0850 & -0.0025 \end{bmatrix}^T$ with $\|T_{\hat{\eta}e\varepsilon}\|_\infty = 9.37$ and $\|T_{\hat{\eta}e\varepsilon}\|_2 = 16$. By solving the resulting IID generator (6) in forward time, the estimated IID is obtained. As shown in Fig.5, the two-dimensional estimated IID $\hat{\eta}$ is bounded, and each element of $y = \dot{\hat{\eta}} - A\hat{\eta} - N\xi \in \mathbb{R}^2$ is bounded ultimately by a very small positive value.

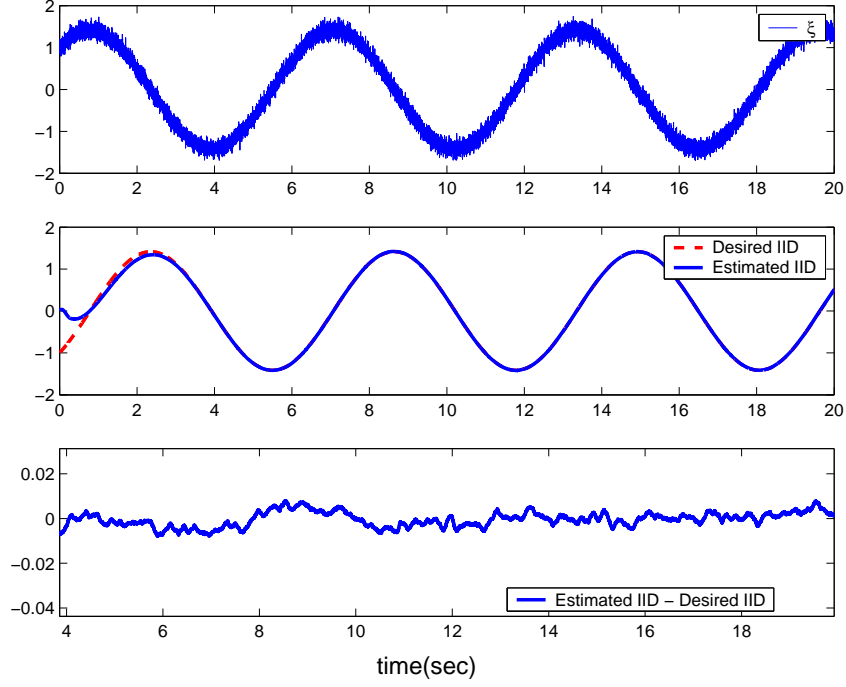


Fig. 4. Output of the IID generator in *Example 1* in the presence of noise

VI. CONCLUSIONS

In this paper, a new causal dynamic IID generator is proposed. By solving it in forward time, the IID can be obtained. Owing to the dynamics, it can suppress noise and perturbations. Compared with the existing similar generators, it is applicable to the singular case and can easily be extended to slowly time-varying unstable matrix differential equations in the same framework without extra computation. The simulation examples demonstrate the effectiveness of the proposed IID generator.

VII. APPENDIX

A. Proof of Lemma 1

Before presenting the proof, we introduce a lemma.

Lemma 2 (PBH controllability test) [16, Theorem 4.8, p.102]. The matrix pair (F, B) is controllable if and only if

$$\text{rank} \begin{bmatrix} F - \lambda I & B \end{bmatrix} = n$$

for every eigenvalue $\lambda \in \mathbb{C}$ of F .

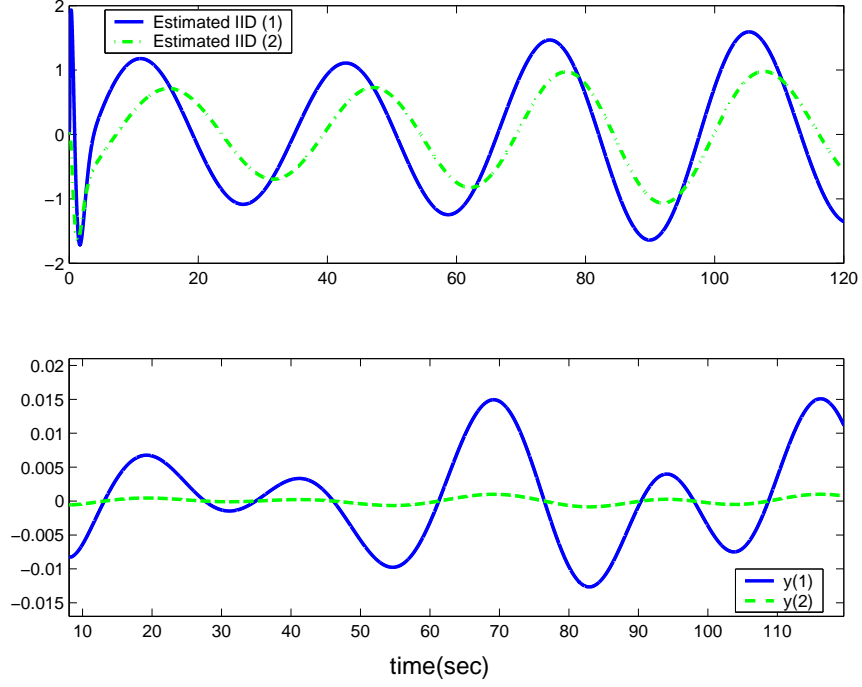


Fig. 5. Output of the IID generator in *Example 2* with a time-varying matrix

Sufficiency of Lemma 1 (A Constructive Proof). For $F \in \mathbb{R}^{n \times n}$, there exists a matrix $T \in \mathbb{R}^{n \times n}$ such that [17, Theorem 9.22, pp.82-83]

$$T^{-1}FT = J = \text{diag}(J_1, \dots, J_{n_s})$$

where each of the Jordan block matrices J_1, \dots, J_{n_s} is of the form

$$J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix} \quad (13)$$

in the case of real eigenvalues λ_i , and

$$J_i = \begin{bmatrix} M_i & I_2 & & \\ & M_i & \ddots & \\ & & \ddots & I_2 \\ & & & M_i \end{bmatrix} \quad (14)$$

where $M_i = \begin{bmatrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{bmatrix}$ and $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ in the case of $\alpha_i \pm j\beta_i, \beta_i \neq 0$. For simplicity and without loss of generality, we assume that only the last Jordan block J_{n_s} is in the form

of (14). The Jordan block J_i corresponds to a left eigenvector $v_i \in \mathbb{R}^n, i = 1, \dots, n_s - 1$, and J_{n_s} corresponds to a couple of left eigenvectors $v_{n_s} \in \mathbb{C}^n, v_{n_s+1} \in \mathbb{C}^n$. It is easy to see that $v_i^H v_k = 0$ by the form of J , $i \neq k$, except for v_{n_s} and v_{n_s+1} . Every eigenvalue $\lambda_i \in \mathbb{C}$ corresponds to a left eigenvector $0 \neq v_i \in \mathbb{C}^n$ such that $v_i^H J = \lambda_i v_i^H, i = 1, \dots, n_s+1$, which implies that $\bar{v}^H J = \bar{\lambda} \bar{v}^H$. Here \bar{x} represents the element-by-element conjugation of $x \in \mathbb{C}^n$, and x^H represents the conjugate transpose of $x \in \mathbb{C}^n$. Therefore, for a couple of conjugate complex roots, their eigenvectors can be chosen to be conjugate, namely $v_{n_s+1} = \bar{v}_{n_s}$, so that

$$B = T \sum_{i=1}^{n_s+1} v_i \quad (15)$$

is a real vector. Next, we will show that $\text{rank} \begin{bmatrix} F - \lambda I_n & B \end{bmatrix} = n$ for every eigenvalue $\lambda \in \mathbb{C}$ of F . Suppose, to the contrary, that there exists a vector $0 \neq p \in \mathbb{C}^n$ and $\lambda_k \in \mathbb{C}$ such that $p^H \begin{bmatrix} F - \lambda_k I_n & B \end{bmatrix} = 0$, namely

$$p^H (F - \lambda_k I_n) = 0$$

$$p^H B = 0.$$

Furthermore, we have

$$p^H T (J - \lambda_k I_n) = 0 \quad (16)$$

$$p^H T \sum_{i=1}^{n_s+1} v_i = 0. \quad (17)$$

Since $\text{rank}(F - \lambda I_n) = \text{rank}(J - \lambda I_n) = n - 1$ for every eigenvalue of F , each eigenvalue corresponds to exactly one eigenvector. As a result, the equation (16) implies $T^H p = \mu v_k, 0 \neq \mu \in \mathbb{C}$. Furthermore, the equation (17) implies

$$\mu v_k^H v_k = 0, k = 1, \dots, n_s - 1 \quad (18)$$

or

$$\mu v_{n_s}^H (v_{n_s} + \bar{v}_{n_s}) = 0, k = n_s \quad (19)$$

or

$$\mu \bar{v}_{n_s}^H (v_{n_s} + \bar{v}_{n_s}) = 0, k = n_s + 1 \quad (20)$$

where the orthogonality and $v_{n_s+1} = \bar{v}_{n_s}$ have been utilized. The equation (18) implies that

$$v_k = 0$$

which contradicts with $v_i \neq 0_n, i = 1, \dots, n_s + 1$. The equation (19) or (20) implies that

$$v_{n_s} + \bar{v}_{n_s} = 0.$$

Consequently, v_{n_s} is in the form of $v_{n_s} = jv_{n_s}^r$, where $v_{n_s}^r \in \mathbb{R}^n$. Since

$$v_{n_s}^H J = (\alpha_{n_s} + j\beta_{n_s}) v_{n_s}^H$$

$$\bar{v}_{n_s}^H J = (\alpha_{n_s} - j\beta_{n_s}) \bar{v}_{n_s}^H$$

we have

$$\begin{aligned} 0 &= (v_{n_s}^H + \bar{v}_{n_s}^H) J \\ &= (\alpha_{n_s} + j\beta_{n_s}) v_{n_s}^H + (\alpha_{n_s} - j\beta_{n_s}) \bar{v}_{n_s}^H \\ &= 2\beta_{n_s} v_{n_s}^r j \end{aligned}$$

which contradicts with $\beta_{n_s} \neq 0$ or $v_{n_s} \neq 0_n$. Therefore, $\text{rank} \begin{bmatrix} F - \lambda I_n & B \end{bmatrix} = n$ for every eigenvalue $\lambda \in \mathbb{C}$ of F , namely the pair (F, B) is controllable by *Lemma 2*.

Necessity of Lemma 1. If $\text{rank}(F - \lambda I_n) \neq n - 1$, namely $\text{rank}(F - \lambda I_n) \leq n - 2$ for every eigenvalue λ of F , then $\text{rank} \begin{bmatrix} F - \lambda I_n & B \end{bmatrix} \leq n - 1$ for any $B \in \mathbb{R}^n$, namely the pair (F, B) is uncontrollable by *Lemma 2*.

B. Proof of Theorem 1

Before proving, we introduce a lemma.

Lemma 3. If the pair (F, B) is controllable, then there exists a vector $C \in \mathbb{R}^n$ such that

$$C^T (sI_n - F)^{-1} B = \frac{1}{\det(sI_n - F)}$$

where $F \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^n$.

Proof. First, we have $(sI_n - F)^{-1} B = G \begin{bmatrix} s^{n-1} & \dots & 1 \end{bmatrix}^T / \det(sI_n - F)$, where $G \in \mathbb{R}^{n \times n}$. If the pair (F, B) is controllable, then the matrix G is of full rank [18]. We can complete this proof by choosing $C = (G^{-1})^T \begin{bmatrix} 0 & \dots & 0 & 1 \end{bmatrix}^T$. \square

Proof of Theorem 1. The IID generator (4) contains the dynamics $\dot{v} = Sv + L_{11}e$. Its Laplace transformation is

$$v(s) = (sI_m - S)^{-1} L_{11}e(s).$$

The condition $\max \text{Re } \lambda(A_{cl}) < 0$ implies that the pair (S, L_{11}) is controllable. Further by *Lemma 3*, there exists a vector $C_e \in \mathbb{R}^m$ such that

$$C_e^T v(s) = C_e^T (sI_m - S)^{-1} L_{11}e(s) = \frac{1}{\det(sI_m - S)} e(s)$$

namely,

$$e(s) = \det(sI_m - S) C_e^T v(s). \quad (21)$$

By (4), the transfer function from ξ to v is

$$v(s) = C_v^T (sI_{m+n+1} - A_{cl})^{-1} N_{cl} \xi(s)$$

where $C_v = \begin{bmatrix} I_m & 0_{m \times n} & 0 \end{bmatrix}^T$. Substituting the equation above into (21) yields

$$e(s) = \det(sI_m - S) C_e^T C_v^T (sI_{m+n+1} - A_{cl})^{-1} N_{cl} \xi(s).$$

Since ξ is generated by (3), we have $\xi(s) = E^T (sI_m - S)^{-1} w(0)$, where $w(0) \in \mathbb{R}^m$.

Since $(sI_m - S)^{-1} = \frac{1}{\det(sI_m - S)} \text{adj}(sI_m - S)$, $e(s)$ is further represented as

$$\begin{aligned} e(s) &= \det(sI_m - S) C_e^T C_v^T (sI_{m+n+1} - A_{cl})^{-1} N_{cl} E^T \frac{1}{\det(sI_m - S)} \text{adj}(sI_m - S) w(0) \\ &= C_e^T C_v^T (sI_{m+n+1} - A_{cl})^{-1} N_{cl} E^T \text{adj}(sI_m - S) w(0). \end{aligned} \quad (22)$$

Since $\max \text{Re } \lambda(A_{cl}) < 0$ and the order of A_{cl} is higher than that of S , for any initial values $w(0)$, we have $e \rightarrow 0$ as $t \rightarrow \infty$ from (22). Since ξ is bounded on $[0, \infty)$ and $\max \text{Re } \lambda(A_{cl}) < 0$, the signals v and $\hat{\eta}$ in (4) are bounded. Since the IID (4) contains the relation $\dot{\hat{\eta}} = A\hat{\eta} + L_{12}e + N\xi$. By the obtained result that $e \rightarrow 0$ as $t \rightarrow \infty$, we have $y = L_{12}e = \dot{\hat{\eta}} - A\hat{\eta} - N\xi \rightarrow 0$ as $t \rightarrow \infty$.

C. Proof of Theorem 3

By condition ii) and Theorem 2, there exists a vector L_1 such that the pair (A_S, L_1) is controllable. Consider the pair

$$\left(\begin{pmatrix} A_S & L_1 \\ 0_{1 \times (n+m)} & 0 \end{pmatrix}, \begin{pmatrix} 0_{n+m} \\ 1 \end{pmatrix} \right). \quad (23)$$

The controllability matrix of pair (23) is

$$W = \begin{pmatrix} 0_{n+m} & L_1 & A_S L_1 & \cdots & A_S^{n+m-1} L_1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Since the pair (A_S, L_1) is controllable, $\text{rank} \begin{pmatrix} 0_{n+m} & L_1 & A_S L_1 & \cdots & A_S^{n+m-1} L_1 \end{pmatrix} = n+m$.

Consequently, $\text{rank} W = n + m + 1$. Therefore, the pair (23) is controllable, namely there must exist gains L_{21}, L_{22}, L_3 such that $\max \text{Re } \lambda(A_{cl}) < 0$. The remainder of proof is the Due to as Theorem 1.

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